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BUCKLING OF THICK ORTHOTROPIC CYLINDRICAL SHELLS UNDER EXTERNAL PRESSURE BASED ON NON-PLANAR EQUILIBRIUM MODES

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Abstract—A formulation based on the three dimensional theory of elasticity is employed to study the buckling of an orthotropic cylindrical shell under external pressure. In this paper, a non-zero axial displacement and a full dependence of the buckling modes on the three coordinates is assumed, as opposed to the ring approximation employed in the earlier studies. The results from this elasticity solution are compared with the critical loads predicted by the orthotropic Donnell and Timoshenko non-shallow shell formulations. Two cases of end conditions are considered; one with both ends of the shell fixed, and the other with both ends capped and under the action of the external pressure. Moreover, two cases of orthotropic material are considered with stiffness constants typical of glass/epoxy and graphite/epoxy. For the isotropic material case, the predictions of the simplified (single expression) Donnell and the Flügge and the Danielson and Simmonds theories are also compared. In all cases, the elasticity approach predicts a lower critical load than the shell theories, the percentage reduction being larger with increasing thickness. The degree of non-conservatism depends strongly on the material properties, being smaller for the isotropic case. Furthermore, although it is a commonly accepted notion that the critical point in loading under external pressure occurs for $n = 2$ and $m = 1$ (number of circumferential waves and number of axial half-waves, respectively), it was found that this is not the case for the strongly orthotropic graphite/epoxy material and the moderately thick construction; for this case, the value of m at the critical point is greater than 1 (yet, in all cases $n = 2$).

1. INTRODUCTION

Shell structural configurations of moderate thickness can be potentially used in the marine industry for submersible hulls as well as for components in the automobile and aircraft industries. Moreover, composites in the form of circular cylindrical shells are considered for civil engineering, column-type applications and in space vehicles as a primary load carrying structure.

In all these applications, an important design parameter is the buckling strength. This is particularly significant in applications involving advanced composites because of the large strength-to-weight ratio and the lack of extensive plastic yielding in these materials.

In shells under external pressure, simple, direct expressions for the critical value are available in the literature only for isotropic material (Donnell, 1933; Flügge, 1960; Danielson and Simmonds, 1969). Besides these simple expressions, which are derived by imposing certain shallowness limitations, values of the critical pressure can be found by solving the eigenvalue problem for the set of cylindrical shell equations from the Donnell theory that are not subject to the shallowness limitations of the simple expressions (Brush and Almroth, 1975). Furthermore, in presenting a shell theory formulation for isotropic shells, Timoshenko and Gere (1961) included some additional terms (these equations are briefly described in the Appendix). Both the (non-simplified) Donnell and Timoshenko shell theory equations can be easily extended for the case of orthotropic material. Although several other shell theories based on the classical hypotheses have been formulated, the Donnell, Timoshenko, Flügge and Danielson and Simmonds theories constitute the representative set of classical shell theories that will be used in this paper for comparing with the results from the benchmark elasticity solution.

Furthermore, although the classical shell theories have been most widely used in deriving critical loads (e.g. Simitses *et al.*, 1985), the recent, higher order, shear deformation theories (e.g. Whitney and Sun, 1974; Librescu, 1975; Reddy and Liu, 1985) could potentially produce much more accurate results. Therefore, a benchmark elasticity solution is needed in order to enable a future comparison of the accuracy of the predictions from the improved shell theories. The anisotropy and the large extensional-to-shear modulus ratio of advanced composites underscores further the need for accurate predictions.

Elasticity solutions for the buckling of cylindrical shells have been recently presented by Kardomateas (1993a) for the case of uniform external pressure and orthotropic material; a simplified problem definition was used in this study ("ring" assumption), in that the pre-buckling stress and displacement field was axisymmetric, and the buckling modes were assumed two dimensional, i.e. no z component of the displacement field and no z-dependence of the r and θ displacement components. It was shown that the critical load for external pressure loading, as predicted by shell theory, can be highly non-conservative for moderately thick construction.

A more thorough investigation of the thickness effects was conducted by Kardomateas (1993b) for the case of a transversely isotropic thick cylindrical shell under axial compression. This work also included a comprehensive study of the performance of the Donnell (1933), the Flügge (1960) and the Danielson and Simmonds (1969) theories for isotropic material in the case of axial compression. These theories were all found to be non-conservative in predicting bifurcation points, the Donnell theory being the most non-conservative.

In a further study, Kardomateas (1993c) considered a generally cylindrically orthotropic material under axial compression. In addition to considering general orthotropy for the material constitutive behavior, the latter work investigated the performance of another classical formulation, i.e. the Timoshenko and Gere (1961) shell theory. The bifurcation points from the Timoshenko formulation were found to be closer to the elasticity predictions than the ones from the Donnell formulation. More importantly, the Timoshenko bifurcation point for the case of pure axial compression was always lower than the elasticity one, i.e. the Timoshenko formulation was conservative. This case of pure axial load from the Timoshenko formulation was actually the only case to date of a classical shell theory rendering conservative estimates of the critical load (the case of combined lateral pressure and axial load has not yet been studied). However, as will be seen in this paper, the same formulation for the case of a shell under external pressure would render non-conservative estimates.

In this paper, a benchmark solution for the buckling of an orthotropic cylindrical shell under external pressure is produced. The non-linear three dimensional theory of elasticity is appropriately formulated and reduced to a standard eigenvalue problem for ordinary linear differential equations in terms of a single variable (the radial distance r), with the applied external pressure, p, the parameter. A full dependence on r, θ and z of the buckling modes is assumed. The formulation employs the exact elasticity solution by Lekhnitskii (1963) for the pre-buckling state. Two cases of end conditions are considered; one with both ends of the shell fixed, which leads to a much easier derivation of the pre-buckling stress field, and the other with both ends capped and under the action of the external pressure.

Results will be presented for the critical load and the buckling modes; these will be compared with both the orthotropic "non-shallow" Donnell and Timoshenko shell formulations. For the isotropic case, a comparison with the simplified Donnell (1933), the Flügge (1960) and the Danielson and Simmonds (1969) formulas will also be performed. The orthotropic material examples are for stiffness constants typical of glass/epoxy and graphite/epoxy and the reinforcing direction along the periphery.

2. FORMULATION

Let us consider the equations of equilibrium in terms of the second Piola–Kirchhoff stress tensor $\boldsymbol{\Sigma}$ in the form

$$\operatorname{div}(\boldsymbol{\Sigma}\cdot\mathbf{F}^{\mathrm{T}}) = 0, \tag{1a}$$

where **F** is the deformation gradient defined by

$$\mathbf{F} = \mathbf{I} + \operatorname{grad}\vec{V}, \tag{1b}$$

where $\vec{V}$ is the displacement vector and **I** is the identity tensor. Notice that the second Piola–Kirchhoff stress tensor is symmetric whenever the Cauchy stress tensor is, and therefore it has been preferred in finite-strain elasticity formulations. Furthermore, because it is symmetric, it can be used on constitutive equations with a symmetric strain tensor.

The strain tensor is defined by

$$\mathbf{E} = \tfrac{1}{2}(\mathbf{F}^{\mathrm{T}}\cdot\mathbf{F}-\mathbf{I}). \tag{1c}$$

More specifically, in terms of the linear strains:

$$e_{rr} = \frac{\partial u}{\partial r}, \quad e_{\theta\theta} = \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r}, \quad e_{zz} = \frac{\partial w}{\partial z}, \tag{2a}$$

$$e_{r\theta} = \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r}, \quad e_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}, \quad e_{\theta z} = \frac{\partial v}{\partial z} + \frac{1}{r}\frac{\partial w}{\partial \theta}, \tag{2b}$$

and the linear rotations:

$$2\omega_r = \frac{1}{r}\frac{\partial w}{\partial \theta} - \frac{\partial v}{\partial z}, \quad 2\omega_\theta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial r}, \quad 2\omega_z = \frac{\partial v}{\partial r} + \frac{v}{r} - \frac{1}{r}\frac{\partial u}{\partial \theta}, \tag{2c}$$

the deformation gradient **F** is

$$\mathbf{F} = \begin{bmatrix} 1+e_{rr} & \frac{1}{2}e_{r\theta}-\omega_z & \frac{1}{2}e_{rz}+\omega_\theta \\ \frac{1}{2}e_{r\theta}+\omega_z & 1+e_{\theta\theta} & \frac{1}{2}e_{\theta z}-\omega_r \\ \frac{1}{2}e_{rz}-\omega_\theta & \frac{1}{2}e_{\theta z}+\omega_r & 1+e_{zz} \end{bmatrix}. \tag{3}$$

At the critical load there are two possible infinitely close positions of equilibrium. The r, θ and z components of the displacement corresponding to the primary position are denoted by u_0, v_0, w_0. A perturbed position is denoted by

$$u = u_0 + \alpha u_1; \quad v = v_0 + \alpha v_1; \quad w = w_0 + \alpha w_1, \tag{4}$$

where α is an infinitesimally small quantity. Here, $\alpha u_1(r,\theta,z)$, $\alpha v_1(r,\theta,z)$, $\alpha w_1(r,\theta,z)$ are the displacements to which the points of the body must be subjected to shift them from the initial position of equilibrium to the new equilibrium position. The functions $u_1(r,\theta,z)$, $v_1(r,\theta,z)$, $w_1(r,\theta,z)$ are assumed finite and α is an infinitesimally small quantity independent of r, θ, z.

Following Kardomateas (1993a), we obtain the following buckling equations:

$$\frac{\partial}{\partial r}(\sigma'_{rr} - \tau^0_{r\theta}\omega'_z + \tau^0_{rz}\omega'_\theta) + \frac{1}{r}\frac{\partial}{\partial \theta}(\tau'_{r\theta} - \sigma^0_{\theta\theta}\omega'_z + \tau^0_{\theta z}\omega'_\theta)$$
$$+ \frac{\partial}{\partial z}(\tau'_{rz} - \tau^0_{\theta z}\omega'_z + \sigma^0_{zz}\omega'_\theta) + \frac{1}{r}(\sigma'_{rr} - \sigma'_{\theta\theta} + \tau^0_{rz}\omega'_\theta + \tau^0_{\theta z}\omega'_r - 2\tau^0_{r\theta}\omega'_z) = 0, \tag{5a}$$

$$\frac{\partial}{\partial r}(\tau'_{r\theta}+\sigma^0_{rr}\omega'_z-\tau^0_{rz}\omega'_r)+\frac{1}{r}\frac{\partial}{\partial\theta}(\sigma'_{\theta\theta}+\tau^0_{r\theta}\omega'_z-\tau^0_{\theta z}\omega'_r)$$
$$+\frac{\partial}{\partial z}(\tau'_{\theta z}+\tau^0_{rz}\omega'_z-\sigma^0_{zz}\omega'_r)+\frac{1}{r}(2\tau'_{r\theta}+\sigma^0_{rr}\omega'_z-\sigma^0_{\theta\theta}\omega'_z+\tau^0_{\theta z}\omega'_\theta-\tau^0_{rz}\omega'_r)=0, \quad (5b)$$

$$\frac{\partial}{\partial r}(\tau'_{rz}-\sigma^0_{rr}\omega'_\theta+\tau^0_{r\theta}\omega'_r)+\frac{1}{r}\frac{\partial}{\partial\theta}(\tau'_{\theta z}-\tau^0_{r\theta}\omega'_\theta+\sigma^0_{\theta\theta}\omega'_r)$$
$$+\frac{\partial}{\partial z}(\sigma'_{zz}-\tau^0_{rz}\omega'_\theta+\tau^0_{\theta z}\omega'_r)+\frac{1}{r}(\tau'_{rz}-\sigma^0_{rr}\omega'_\theta+\tau^0_{r\theta}\omega'_r)=0. \quad (5c)$$

In the previous equations, σ^0_{ij} and ω^0_j are the values of σ_{ij} and ω_j at the initial equilibrium position, i.e. for $u=u_0$, $v=v_0$ and $w=w_0$, and σ'_{ij} and ω'_j are the values at the perturbed position, i.e. for $u=u_1$, $v=v_1$ and $w=w_1$.

The boundary conditions associated with 1(a) can be expressed as:

$$(\mathbf{F}\cdot\mathbf{\Sigma}^{\mathrm{T}})\cdot\hat{N}=\vec{t}(\vec{V}), \quad (6)$$

where $\vec{t}$ is the traction vector on the surface which has outward unit normal $\hat{N}=(\hat{l},\hat{m},\hat{n})$ before any deformation. The traction vector $\vec{t}$ depends on the displacement field $\vec{V}=(u,v,w)$. Again, following Kardomateas (1993a), we obtain for the lateral and end surfaces:

$$(\sigma'_{rr}-\tau^0_{r\theta}\omega'_z+\tau^0_{rz}\omega'_\theta)\hat{l}+(\tau'_{r\theta}-\sigma^0_{\theta\theta}\omega'_z+\tau^0_{\theta z}\omega'_\theta)\hat{m}+(\tau'_{rz}-\tau^0_{\theta z}\omega'_z+\sigma^0_{zz}\omega'_\theta)\hat{n}=p(\omega'_z\hat{m}-\omega'_\theta\hat{n}), \quad (7a)$$

$$(\tau'_{r\theta}+\sigma^0_{rr}\omega'_z-\tau^0_{rz}\omega'_r)\hat{l}+(\sigma'_{\theta\theta}+\tau^0_{r\theta}\omega'_z-\tau^0_{\theta z}\omega'_r)\hat{m}+(\tau'_{\theta z}+\tau^0_{rz}\omega'_z-\sigma^0_{zz}\omega'_r)\hat{n}=-p(\omega'_z\hat{l}-\omega'_r\hat{n}), \quad (7b)$$

$$(\tau'_{rz}+\tau^0_{r\theta}\omega'_r-\sigma^0_{rr}\omega'_\theta)\hat{l}+(\tau'_{\theta z}+\sigma^0_{\theta\theta}\omega'_r-\tau^0_{r\theta}\omega'_\theta)\hat{m}+(\sigma'_{zz}+\tau^0_{\theta z}\omega'_r-\tau^0_{rz}\omega'_\theta)\hat{n}=p(\omega'_\theta\hat{l}-\omega'_r\hat{m}). \quad (7c)$$

2.1. *Pre-buckling state*

The problem under consideration is that of an orthotropic cylindrical shell subjected to a uniform external pressure, p. Two cases will be considered; one where both ends of the shell are fixed (this simplifies the derivation of the pre-buckling stress field), and the other where the ends are capped and under the action of the external pressure, p (this would more closely resemble the state of loading in a submersible). The stress–strain relations for the orthotropic body are

$$\begin{bmatrix}\sigma_{rr}\\ \sigma_{\theta\theta}\\ \sigma_{zz}\\ \tau_{\theta z}\\ \tau_{rz}\\ \tau_{r\theta}\end{bmatrix}=\begin{bmatrix}c_{11}&c_{12}&c_{13}&0&0&0\\ c_{12}&c_{22}&c_{23}&0&0&0\\ c_{13}&c_{23}&c_{33}&0&0&0\\ 0&0&0&c_{44}&0&0\\ 0&0&0&0&c_{55}&0\\ 0&0&0&0&0&c_{66}\end{bmatrix}\begin{bmatrix}\varepsilon_{rr}\\ \varepsilon_{\theta\theta}\\ \varepsilon_{zz}\\ \gamma_{\theta z}\\ \gamma_{rz}\\ \gamma_{r\theta}\end{bmatrix}, \quad (8a)$$

where c_{ij} $(i,j=1,2,3)$ are the stiffness constants (we have used the notation $1\equiv r$, $2\equiv\theta$, $3\equiv z$).

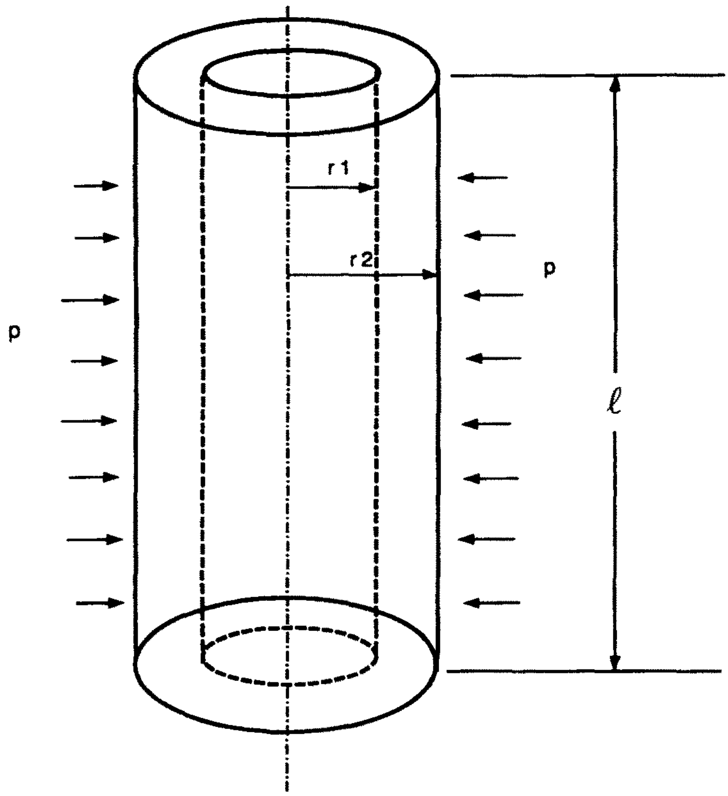

Fig. 1. Cylindrical shell under external pressure.

Let r_1 be the internal and r_2 the external radius (Fig. 1) and $c = r_1/r_2$. In terms of

$$k = \sqrt{\frac{c_{22}}{c_{11}}}, \tag{8b}$$

the stress field for the simpler case of a cylinder with both ends fixed is given directly from Lekhnitskii (1963) as follows:

$$\sigma_{rr}^0 = p(C_1 r^{k-1} + C_2 r^{-k-1}), \tag{9a}$$

$$\sigma_{\theta\theta}^0 = p(C_1 k r^{k-1} - C_2 k r^{-k-1}), \tag{9b}$$

$$\sigma_{zz}^0 = -p\left(C_1 \frac{a_{13}+ka_{23}}{a_{33}} r^{k-1} + C_2 \frac{a_{13}-ka_{23}}{a_{33}} r^{-k-1}\right), \tag{9c}$$

$$\tau_{r\theta}^0 = \tau_{rz}^0 = \tau_{\theta z}^0 = 0, \tag{9d}$$

where

$$C_1 = -\frac{1}{(1-c^{2k})r_2^{k-1}}; \quad C_2 = \frac{c^{2k} r_2^{k+1}}{(1-c^{2k})}. \tag{9e}$$

In the previous equations a_{ij} are the compliance constants, i.e.

$$\begin{bmatrix} \varepsilon_{rr} \\ \varepsilon_{\theta\theta} \\ \varepsilon_{zz} \\ \gamma_{\theta z} \\ \gamma_{rz} \\ \gamma_{r\theta} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 & 0 \\ a_{12} & a_{22} & a_{23} & 0 & 0 & 0 \\ a_{13} & a_{23} & a_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & a_{66} \end{bmatrix} \begin{bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \sigma_{zz} \\ \tau_{\theta z} \\ \tau_{rz} \\ \tau_{r\theta} \end{bmatrix}. \tag{10}$$

For the case of a shell with end caps under the action of the external pressure, the stresses that satisfy the equilibrium equations in the pre-buckling state, arise from a displacement field accompanied by deformation (assume $c_{11} \neq c_{22}$):

$$u_0(r) = C_{10}r^k + C_{20}r^{-k} + \frac{c_{23}-c_{13}}{c_{11}-c_{22}}D_0 r; \quad v_0 = 0; \quad w_0(z) = D_0 z. \tag{11a}$$

Satisfying the inside boundary traction-free condition, $\sigma_{rr}^0|_{r_1} = 0$, allows eliminating D_0 and gives the radial stress in the form

$$\sigma_{rr}^0 = C_{10}(c_{11}k + c_{12})(r^{k-1} - r_1^{k-1}) + C_{20}(-c_{11}k + c_{12})(r^{-k-1} - r_1^{-k-1}), \tag{11b}$$

the hoop stress in the form

$$\sigma_{\theta\theta}^0 = C_{10}[(c_{12}k + c_{22})r^{k-1} - (c_{11}k + c_{12})fr_1^{k-1}] + C_{20}[(-c_{12}k + c_{22})r^{-k-1} - (-c_{11}k + c_{12})fr_1^{-k-1}], \tag{11c}$$

and the axial stress in the form

$$\sigma_{zz}^0 = C_{10}[(c_{13}k + c_{23})r^{k-1} - (c_{11}k + c_{12})gr_1^{k-1}] + C_{20}[(-c_{13}k + c_{23})r^{-k-1} - (-c_{11}k + c_{12})gr_1^{-k-1}]. \tag{11d}$$

In the previous relations, f and g are in terms of the stiffness constants:

$$f = \frac{(c_{12}+c_{22})(c_{23}-c_{13}) + c_{23}(c_{11}-c_{22})}{(c_{11}+c_{12})(c_{23}-c_{13}) + c_{13}(c_{11}-c_{22})}; \quad g = \frac{c_{23}^2 - c_{13}^2 + c_{33}(c_{11}-c_{22})}{(c_{11}+c_{12})(c_{23}-c_{13}) + c_{13}(c_{11}-c_{22})}. \tag{11e}$$

The constants C_{10} and C_{20} are linearly dependent on the external pressure and are found from the condition of external pressure

$$\sigma_{rr}^0|_{r_2} = -p, \tag{12a}$$

and the axial force developed due to the pressure on the end caps

$$\int_{r_1}^{r_2} \sigma_{zz}^0 r \,\mathrm{d}r = -\frac{p}{2}r_2^2, \tag{12b}$$

as follows:

$$C_{10} = p\frac{\beta_{12}-\beta_{22}}{\beta_{11}\beta_{22}-\beta_{12}\beta_{21}}; \quad C_{20} = p\frac{\beta_{21}-\beta_{11}}{\beta_{11}\beta_{22}-\beta_{12}\beta_{21}}, \tag{12c}$$

where

$$\beta_{11} = (c_{11}k+c_{12})(r_2^{k-1}-r_1^{k-1}); \quad \beta_{12} = (-c_{11}k+c_{12})(r_2^{-k-1}-r_1^{-k-1}), \tag{12d}$$

$$\beta_{21} = \frac{2(c_{13}k+c_{23})}{(k+1)}\left(\frac{r_2^{k+1}-r_1^{k+1}}{r_2^2}\right)-(c_{11}k+c_{12})gr_1^{k-1}\left(\frac{r_2^2-r_1^2}{r_2^2}\right), \tag{12e}$$

$$\beta_{22} = \frac{2(-c_{13}k+c_{23})}{(1-k)}\left(\frac{r_2^{-k+1}-r_1^{-k+1}}{r_2^2}\right)-(-c_{11}k+c_{12})gr_1^{-k-1}\left(\frac{r_2^2-r_1^2}{r_2^2}\right). \tag{12f}$$

Hence, it turns out that in both cases the pre-buckling shear stresses are zero and the pre-buckling normal stresses are linearly dependent on the external pressure, p, in the form

$$\sigma_{ij}^0 = p(C_{ij,0}+C_{ij,1}r^{k-1}+C_{ij,2}r^{-k-1}), \tag{13}$$

where $C_{ij,0}$, $C_{ij,1}$, $C_{ij,2}$, are constants dependent on the material properties, the geometric dimensions and the circumferential and axial wave numbers n and λ. This observation allows a direct implementation of a standard solution scheme, since, as will be seen, the derivatives of the stresses with respect to p will be needed and these are directly found from eqn (13).

2.2. *Perturbed state*

Using the constitutive relations [eqn (8a)] for the stresses σ'_{ij} in terms of strains e'_{ij}, the strain–displacement relations [eqn (2)] for the strains e'_{ij} and the rotations ω'_j in terms of the displacements u_1, v_1, w_1, and taking into account [eqn (9d)], the buckling eqn (5a) for the problem at hand is written in terms of the displacements at the perturbed state as follows:

$$c_{11}\left(u_{1,rr}+\frac{u_{1,r}}{r}\right)-c_{22}\frac{u_1}{r^2}+\left(c_{66}+\frac{\sigma_{\theta\theta}^0}{2}\right)\frac{u_{1,\theta\theta}}{r^2}+\left(c_{55}+\frac{\sigma_{zz}^0}{2}\right)u_{1,zz}$$
$$+\left(c_{12}+c_{66}-\frac{\sigma_{\theta\theta}^0}{2}\right)\frac{v_{1,r\theta}}{r}-\left(c_{22}+c_{66}+\frac{\sigma_{\theta\theta}^0}{2}\right)\frac{v_{1,\theta}}{r^2}$$
$$+\left(c_{13}+c_{55}-\frac{\sigma_{zz}^0}{2}\right)w_{1,rz}+(c_{13}-c_{23})\frac{w_{1,z}}{r} = 0. \tag{14a}$$

The second buckling eqn (5b) gives:

$$\left(c_{66}+\frac{\sigma_{rr}^0}{2}\right)\left(v_{1,rr}+\frac{v_{1,r}}{r}-\frac{v_1}{r^2}\right)+\left(\frac{\sigma_{rr}^0-\sigma_{\theta\theta}^0}{2}\right)\left(\frac{v_{1,r}}{r}+\frac{v_1}{r^2}\right)+c_{22}\frac{v_{1,\theta\theta}}{r^2}$$
$$+\left(c_{44}+\frac{\sigma_{zz}^0}{2}\right)v_{1,zz}+\left(c_{66}+c_{12}-\frac{\sigma_{rr}^0}{2}\right)\frac{u_{1,r\theta}}{r}+\left(c_{66}+c_{22}+\frac{\sigma_{\theta\theta}^0}{2}\right)\frac{u_{1,\theta}}{r^2}$$
$$+\left(c_{23}+c_{44}-\frac{\sigma_{zz}^0}{2}\right)\frac{w_{1,\theta z}}{r}+\frac{1}{2}\frac{d\sigma_{rr}^0}{dr}\left(v_{1,r}+\frac{v_1}{r}-\frac{u_{1,\theta}}{r}\right) = 0. \tag{14b}$$

In a similar fashion, the third buckling eqn (5c) gives:

$$\left(c_{55}+\frac{\sigma_{rr}^0}{2}\right)\left(w_{1,rr}+\frac{w_{1,r}}{r}\right)+\left(c_{44}+\frac{\sigma_{\theta\theta}^0}{2}\right)\frac{w_{1,\theta\theta}}{r^2}+c_{33}w_{1,zz}$$
$$+\left(c_{13}+c_{55}-\frac{\sigma_{rr}^0}{2}\right)u_{1,rz}+\left(c_{23}+c_{55}-\frac{\sigma_{rr}^0}{2}\right)\frac{u_{1,z}}{r}$$
$$+\left(c_{23}+c_{44}-\frac{\sigma_{\theta\theta}^0}{2}\right)\frac{v_{1,\theta z}}{r}+\frac{1}{2}\frac{d\sigma_{rr}^0}{dr}(w_{1,r}-u_{1,z}) = 0. \tag{14c}$$

In the perturbed position, we seek equilibrium modes in the form:

$$u_1(r,\theta,z) = U(r)\cos n\theta \sin\lambda z\,; \quad v_1(r,\theta,z) = V(r)\sin n\theta\sin\lambda z\,;$$
$$w_1(r,\theta,z) = W(r)\cos n\theta\cos\lambda z, \tag{15}$$

where the functions $U(r)$, $V(r)$, $W(r)$ are uniquely determined for a particular choice of n and λ.

Substituting in eqn (14a), we obtain the following linear homogeneous ordinary differential equation for $r_1 \leqslant r \leqslant r_2$:

$$U(r)''c_{11} + U(r)'\frac{c_{11}}{r} + U(r)\left[-c_{55}\lambda^2 - \frac{c_{22}+c_{66}n^2}{r^2} - \sigma_{zz}^0\frac{\lambda^2}{2} - \sigma_{\theta\theta}^0\frac{n^2}{2r^2}\right]$$
$$+V(r)'\left[\frac{(c_{12}+c_{66})n}{r} - \sigma_{\theta\theta}^0\frac{n}{2r}\right] + V(r)\left[\frac{-(c_{22}+c_{66})n}{r^2} - \sigma_{\theta\theta}^0\frac{n}{2r^2}\right]$$
$$+W(r)'\left[-(c_{13}+c_{55})\lambda + \sigma_{zz}^0\frac{\lambda}{2}\right] + W(r)\frac{(c_{23}-c_{13})\lambda}{r} = 0. \tag{16a}$$

The second differential eqn (14b) gives for $r_1 \leqslant r \leqslant r_2$:

$$V(r)''\left(c_{66}+\frac{\sigma_{rr}^0}{2}\right) + V(r)'\left[\frac{c_{66}}{r} + \frac{1}{r}\left(\sigma_{rr}^0 - \frac{\sigma_{\theta\theta}^0}{2}\right) + \frac{\sigma_{rr}^{0\prime}}{2}\right] + V(r)\left[-c_{44}\lambda^2 - \frac{c_{66}+c_{22}n^2}{r^2}\right.$$
$$\left. -\sigma_{zz}^0\frac{\lambda^2}{2} - \frac{\sigma_{\theta\theta}^0}{2r^2} + \frac{\sigma_{rr}^{0\prime}}{2r}\right] + U(r)'\left[\frac{-(c_{12}+c_{66})n}{r} + \sigma_{rr}^0\frac{n}{2r}\right]$$
$$+U(r)\left[\frac{-(c_{22}+c_{66})n}{r^2} - \sigma_{\theta\theta}^0\frac{n}{2r^2} + \sigma_{rr}^{0\prime}\frac{n}{2r}\right] + W(r)\left[(c_{23}+c_{44})\frac{n\lambda}{r} - \sigma_{zz}^0\frac{n\lambda}{2r}\right] = 0. \tag{16b}$$

In a similar fashion, eqn (14c) gives for $r_1 \leqslant r \leqslant r_2$:

$$W(r)''\left(c_{55}+\frac{\sigma_{rr}^0}{2}\right) + W(r)'\left[\frac{c_{55}}{r} + \frac{\sigma_{rr}^0}{2r} + \frac{\sigma_{rr}^{0\prime}}{2}\right] + W(r)\left[-c_{33}\lambda^2 - c_{44}\frac{n^2}{r^2} - \sigma_{\theta\theta}^0\frac{n^2}{2r^2}\right]$$
$$+U(r)'\left[(c_{13}+c_{55})\lambda - \sigma_{rr}^0\frac{\lambda}{2}\right] + U(r)\left[\frac{(c_{23}+c_{55})\lambda}{r} - \sigma_{rr}^0\frac{\lambda}{2r} - \sigma_{rr}^{0\prime}\frac{\lambda}{2}\right]$$
$$+V(r)\left[(c_{23}+c_{44})\frac{n\lambda}{r} - \sigma_{\theta\theta}^0\frac{n\lambda}{2r}\right] = 0. \tag{16c}$$

All the previous three eqns (16a–c) are linear, homogeneous, ordinary differential equations of the second order for $U(r)$, $V(r)$ and $W(r)$. In these equations, $\sigma_{rr}^0(r)$, $\sigma_{\theta\theta}^0(r)$, $\sigma_{zz}^0(r)$ and $\sigma_{rr}^{0\prime}(r)$ depend linearly on the external pressure p through expressions in the form of eqn (13).

Now we proceed to the boundary conditions on the lateral surfaces $r = r_j$, $j = 1, 2$. These will complete the formulation of the eigenvalue problem for the critical load.

From eqn (7), we obtain for $\hat{l} = \pm 1$, $\hat{m} = \hat{n} = 0$:

$$\sigma'_{rr} = 0\,; \quad \tau'_{r\theta} + (\sigma_{rr}^0 + p_j)\omega'_z = 0\,; \quad \tau'_{rz} - (\sigma_{rr}^0 + p_j)\omega'_\theta = 0, \quad \text{at } r = r_1, r_2 \tag{17}$$

where $p_j = p$ for $j = 2$, i.e. $r = r_2$ (outside boundary) and $p_j = 0$ for $j = 1$, i.e. $r = r_1$ (inside boundary).

Substituting in eqns (8a), (2), (15) and (9d), the boundary condition $\sigma'_{rr} = 0$ at $r = r_j$, $j = 1, 2$ gives:

$$U'(r_j)c_{11}+[U(r_j)+nV(r_j)]\frac{c_{12}}{r_j}-c_{13}\lambda W(r_j) = 0, \quad j = 1,2. \tag{18a}$$

The boundary condition $\tau'_{r\theta}+(\sigma^0_{rr}+p_j)\omega'_z = 0$ at $r = r_j, j = 1, 2$ gives

$$V'(r_j)\left[c_{66}+(\sigma^0_{rr}+p_j)\frac{1}{2}\right]+[V(r_j)+nU(r_j)]\left[-c_{66}+(\sigma^0_{rr}+p_j)\frac{1}{2}\right]\frac{1}{r_j} = 0, \quad j = 1,2. \tag{18b}$$

In a similar fashion, the condition $\tau'_{rz}-(\sigma^0_{rr}+p_j)\omega'_\theta = 0$ at $r = r_j, j = 1, 2$ gives:

$$\lambda U(r_j)[c_{55}-(\sigma^0_{rr}+p_j)\tfrac{1}{2}]+W'(r_j)[c_{55}+(\sigma^0_{rr}+p_j)\tfrac{1}{2}] = 0, \quad j = 1,2. \tag{18c}$$

Equations (16) and (18) constitute an eigenvalue problem for differential equations, with the applied external pressure, p, the parameter, which can be solved by standard numerical methods (two point boundary value problem).

Before discussing the numerical procedure used for solving this eigenvalue problem, one final point will be addressed. To completely satisfy all the elasticity requirements, we should discuss the boundary conditions at the ends. From eqn (7), the boundary conditions on the ends $\hat{l} = \hat{m} = 0, \hat{n} = \pm 1$, are:

$$\tau'_{rz}+(\sigma^0_{zz}+p)\omega'_\theta = 0; \quad \tau'_{\theta z}-(\sigma^0_{zz}+p)\omega'_r = 0; \quad \sigma'_{zz} = 0, \quad \text{at } z = 0, \ell. \tag{19}$$

These conditions are strictly valid for capped ends; for fixed ends, $p = 0$ on the end faces. However the discussion that follows remains the same in either case.

Since σ'_{zz} varies as $\sin\lambda z$, the condition $\sigma'_{zz} = 0$ on both the lower end $z = 0$, and the upper end $z = \ell$, is satisfied if

$$\lambda = \frac{m\pi}{\ell}. \tag{20}$$

It will be proved now that these remaining two conditions are satisfied on average. To show this we write each of the first two expressions in eqn (19) in the form: $S_{rz} = \tau'_{rz}+(\sigma^0_{zz}+p)\omega'_\theta$ and $S_{\theta z} = \tau'_{\theta z}-(\sigma^0_{zz}+p)\omega'_r$, and integrate their resultants in the Cartesian coordinate system (x, y, z) e.g. the x-resultant of S_{rz} is:

$$\int_{r_1}^{r_2}\int_0^{2\pi} S_{rz}(\cos\theta)(r\,\mathrm{d}\theta)\,\mathrm{d}r.$$

Since τ'_{rz} and ω'_θ have the form of $F(r)\cos n\theta\cos\lambda z$, i.e. they have a $\cos n\theta$ variation, the x-component of S_{rz} has a $\cos n\theta\cos\theta$ variation, which, when integrated over the entire angle range from zero to 2π, will result in zero. The y-component has a $\cos n\theta\sin\theta$ variation, which again, when integrated over the entire angle range, will result in zero. Similar arguments hold for $S_{\theta z}$, which has the form of $F(r)\sin n\theta\cos\lambda z$.

Moreover, it can also be proved that for the system of resultant stresses eqn (19) would produce no torsional moment. Indeed, this moment would be given by

$$\int_{r_1}^{r_2}\int_0^{2\pi} S_{\theta z}(r\,\mathrm{d}\theta)r\,\mathrm{d}r.$$

Since $\tau'_{\theta z}$ and ω'_r and hence $S_{\theta z}$ have a $\sin n\theta$ variation, the previous integral will be in the form

$$\int_{r_1}^{r_2}\int_0^{2\pi} r^2F(r)\sin n\theta\cos\lambda z\,\mathrm{d}r\,\mathrm{d}\theta,$$

which, when integrated over the entire θ-range from zero to 2π, will result in zero.

Returning to the discussion of the eigenvalue problem, as has already been stated, eqns (16) and (18) constitute an eigenvalue problem for ordinary second order linear differential equations in the r variable, with the applied external pressure, p, the parameter. This is essentially a standard two point boundary value problem. The relaxation method was used (Press *et al.*, 1989) which is essentially based on replacing the system of ordinary differential equations by a set of finite difference equations on a grid of points that spans the entire thickness of the shell. For this purpose, an equally spaced mesh of 241 points was employed and the procedure turned out to be highly efficient with rapid convergence. As an initial guess for the iteration process, the shell theory solution was used. An investigation of the convergence showed that the solution converged monotonically and that with even three times as many mesh points, the results differed by less than 0.005 per cent. The procedure employs the derivatives of the equations with respect to the functions U, V, W, U', V', W' and the pressure p; hence, because of the linear nature of the equations and the linear dependence of σ_{ij}^0 on p through eqn (13), it can be directly implemented. Finally, it should be noted that finding the critical load involves a minimization step in the sense that the eigenvalue is obtained for different combinations of n, m and the critical load is the minimum. The values of $n = 2$, $m = 1$ were found to give the minimum eigenvalue in most but not all the cases studied. The specific results are presented in the following.

3. DISCUSSION OF RESULTS

Results for the critical pressure, normalized in terms of the shell thickness, h, as

$$\tilde{p} = \frac{pr_2^3}{E_2h^3}, \tag{21}$$

were produced for a typical glass/epoxy material with moduli in GN/m^2 and Poisson's ratios listed below, where 1 is the radial (r), 2 is the circumferential (θ), and 3 the axial (z) direction: $E_1 = 14.0$, $E_2 = 57.0$, $E_3 = 14.0$, $G_{12} = 5.7$, $G_{23} = 5.7$, $G_{31} = 5.0$, $\nu_{12} = 0.068$, $\nu_{23} = 0.277$, $\nu_{31} = 0.400$. It has been assumed that the reinforcing direction is along the periphery.

In the shell theory solutions, the radial displacement is constant through the thickness and the axial and circumferential ones have a linear variation, i.e. they are in the form

$$u_1(r,\theta,z) = U_0\cos n\theta\sin\lambda z,\quad v_1(r,\theta,z) = \left[V_0+\frac{r-R}{R}(V_0+nU_0)\right]\sin n\theta\sin\lambda z, \tag{22a}$$

$$w_1(r,\theta,z) = [W_0-(r-R)\lambda U_0]\cos n\theta\cos\lambda z, \tag{22b}$$

where $R = (r_1+r_2)/2$ is the mean shell radius and U_0, V_0, W_0 are constants (these displacement field variations would satisfy the classical assumptions of $e_{rr} = e_{r\theta} = e_{rz} = 0$).

A distinct eigenvalue corresponds to each pair of the positive integers m and n. The pair corresponding to the smallest eigenvalue can be determined by trial. As noted in the Introduction, one of the classical theories that will be used for comparison purposes is the "non-shallow" Donnell shell theory formulation. The other benchmark shell theory used in this paper is the one described in Timoshenko and Gere (1961). In this theory, an additional term in the first equation, namely, $-N_\theta^0(v_{,\theta z}+u_{,z})$, and an additional term in the second equation, namely, $RN_z^0v_{,zz}$, exist (these equations together with the extra terms are explicitly given in the Appendix).

Table 1. Comparison with shell theories for glass/epoxy

r_2/r_1	Elasticity	Donnell shell* (% increase)	Timoshenko shell* (% increase)
1.05	0.2813	0.2926 (4.0%)	0.2914 (3.6%)
1.10	0.2744	0.2973 (8.3%)	0.2962 (7.9%)
1.15	0.2758	0.3133 (13.6%)	0.3122 (13.2%)
1.20	0.2764	0.3308 (19.7%)	0.3296 (19.2%)
1.25	0.2755	0.3485 (26.5%)	0.3473 (26.1%)
1.30	0.2733	0.3662 (34.0%)	0.3649 (33.5%)

* See Appendix.
Orthotropic with circumferential reinforcement, $\ell/r_2 = 10$; Critical pressure, $\bar{p} = pr_2^3/(E_2h^3)$; Moduli in GN/m^2: $E_2 = 57$, $E_1 = E_3 = 14$, $G_{31} = 5.0$, $G_{12} = G_{23} = 5.7$; Poisson's ratios: $\nu_{12} = 0.068$, $\nu_{23} = 0.277$, $\nu_{31} = 0.400$; Capped ends, $n = 2$, $m = 1$.

In the comparison studies we have used an extension of the original, isotropic Donnell and Timoshenko formulations for the case of orthotropy. The linear algebraic equations for the eigenvalues of both the Donnell and Timoshenko theories are given in more detail in the Appendix.

Concerning the present elasticity formulation, the critical load is obtained by finding the solution p for a range of n and m and keeping the minimum value. Tables 1 and 2 show the critical pressure, as predicted by the present three dimensional elasticity formulation and the one, as predicted by both the "non-shallow" Donnell and Timoshenko shell equations for the glass/epoxy and graphite/epoxy material, respectively (case of capped ends under pressure). A length ratio $\ell/r_2 = 10$ has been assumed. A range of outside versus inside radius, r_2/r_1 from somewhat thin (1.05) to thick (1.30) is examined. The following observations can be made:

(1) For both the orthotropic material cases, both the Donnell and the Timoshenko bifurcation points are always higher than the elasticity solution, which means that both shell theories are non-conservative. Moreover, they become more non-conservative with thicker construction. Notice that the result for the Timoshenko theory in this case of a shell under external pressure is opposite to the one for a shell under pure axial load, in which case the Timoshenko shell theory was found to be conservative (Kardomateas, 1993c).

(2) Although it is a commonly accepted notion that the critical point in loading under external pressure occurs for $n = 2$ and $m = 1$, it was found that this is not the case for the strongly orthotropic graphite/epoxy material and the moderately thick construction (Table 2); for this case, the value of m at the critical point is greater than 1. However, in all cases $n = 2$.

(3) The bifurcation points from the Timoshenko formulation are always slightly closer to the elasticity predictions than the ones from the Donnell formulation.

(4) The degree of non-conservatism is strongly dependent on the material; the shell theories predict much higher deviations from the elasticity solution for the graphite/epoxy (which is also noted to have a much higher extensional-to-shear modulus ratio).

Table 2. Comparison with shell theories for graphite/epoxy

r_2/r_1	Elasticity (n, m)	Donnell shell* (n, m) (% increase)	Timoshenko shell* (n, m) (% increase)
1.05	0.2576 (2, 1)	0.2723 (2, 1) (5.7%)	0.2713 (2, 1) (5.3%)
1.10	0.2513 (2, 1)	0.2871 (2, 1) (14.2%)	0.2861 (2, 1) (13.8%)
1.15	0.2347 (2, 2)	0.3037 (2, 2) (29.4%)	0.2995 (2, 2) (27.6%)
1.20	0.2166 (2, 3)	0.3183 (2, 2) (47.0%)	0.3111 (2, 3) (43.6%)
1.25	0.1978 (2, 3)	0.3310 (2, 3) (67.3%)	0.3198 (2, 4) (61.7%)
1.30	0.1808 (2, 4)	0.3429 (2, 4) (89.7%)	0.3261 (2, 5) (80.4%)

* See Appendix.
Orthotropic with circumferential reinforcement, $\ell/r_2 = 10$; Critical pressure, $\bar{p} = pr_2^3/(E_2h^3)$; Moduli in GN/m^2: $E_2 = 140$, $E_1 = 9.9$, $E_3 = 9.1$, $G_{31} = 5.9$, $G_{12} = 4.7$, $G_{23} = 4.3$; Poisson's ratios: $\nu_{12} = 0.020$, $\nu_{23} = 0.300$, $\nu_{31} = 0.490$; Capped ends.

Table 3 gives the predictions of the Donnell and Timoshenko shell theories for the glass/epoxy material, in comparison with the elasticity one for the case of fixed ends. A comparison with Table 1 reveals that the end conditions (fixed ends versus capped under pressure ends) have little influence on the critical load. However, two observations can be easily made; the bifurcation load for the capped ends is always slightly smaller than the one for the fixed ends, and the Timoshenko bifurcation point is almost identical to the one for the Donnell point for fixed ends, unlike for capped ends. Hence, it can be concluded that the additional term in the second shell theory equation, namely, $RN_z^0 v_{,zz}$ (which would be zero for fixed ends) is primarily responsible for the differences in the two shell theories and also for the conservatism of the Timoshenko shell theory when pure axial loading is considered. Notice that the study in Kardomateas (1993b) did not include a comparison with the Timoshenko's shell theory.

Particularly simple formulas can be obtained for isotropic materials. Set

$$\tilde{m} = \frac{m\pi R}{\ell}. \tag{23}$$

With some additional shallowness assumptions, a direct formula can be obtained from the Donnell shell theory, in terms of the Young's modulus, E, and the Poisson's ratio, ν, as follows:

$$p_{S-\text{Donnell}} = \frac{Eh}{R}\left[\frac{h^2}{12R^2(1-\nu^2)}\frac{(\tilde{m}^2+n^2)^2}{n^2}+\frac{\tilde{m}^4}{n^2(\tilde{m}^2+n^2)^2}\right]. \tag{24a}$$

For isotropic materials two other shell theories, namely the Flügge (1960) and the Danielson and Simmonds (1969), have produced direct results for the critical external pressure in shells and should, therefore, be compared with the present elasticity solution. The expression for the eigenvalues derived from the Flügge equations (Flügge, 1960), p_F, and the more simplified, but just as accurate, one by Danielson and Simmonds (1969), p_{DS}, are:

$$p_{\{F,DS\}} = \frac{Eh}{R}\frac{Q_{F,DS}}{n^2[(\tilde{m}^2+n^2)^2-(3\tilde{m}^2+n^2)]}, \tag{24b}$$

where the numerator for the Flügge theory is

$$Q_F = \frac{h^2}{12R^2(1-\nu^2)}\{(\tilde{m}^2+n^2)^4-2[\nu m^2+3\tilde{m}^4 n^2 + (4-\nu)\tilde{m}^2 n^4+n^6]+2(2-\nu)\tilde{m}^2 n^2+n^4\}+\tilde{m}^4, \tag{24c}$$

and for the Danielson and Simmonds equations

Table 3. Comparison with shell theories for glass/epoxy-fixed ends

r_2/r_1	Elasticity	Donnell shell* (% increase)	Timoshenko shell* (% increase)
1.05	0.2860	0.2972 (3.9%)	0.2972 (3.9%)
1.10	0.2789	0.3017 (8.2%)	0.3017 (8.2%)
1.15	0.2803	0.3178 (13.4%)	0.3178 (13.4%)
1.20	0.2808	0.3354 (19.4%)	0.3353 (19.4%)
1.25	0.2798	0.3532 (26.2%)	0.3531 (26.2%)
1.30	0.2776	0.3709 (33.6%)	0.3708 (33.6%)

* See Appendix.

Orthotropic with circumferential reinforcement, $\ell/r_2 = 10$; Critical pressure, $\bar{p} = pr_2^3/(E_2h^3)$; Moduli in GN/m^2: $E_2 = 57$, $E_1 = E_3 = 14$, $G_{31} = 5.0$, $G_{12} = G_{23} = 5.7$; Poisson's ratios: $\nu_{12} = 0.068$, $\nu_{23} = 0.277$; $\nu_{31} = 0.400$; Fixed ends, $n = 2$, $m = 1$.

Table 4. Comparison with shell theories for isotropic material

r_2/r_1	Elasticity	Donnell*	Timoshenko*	Simplified Donnell†	Flügge†	Danielson & Simmonds†
1.05	0.3759	0.3907 (3.9%)	0.3906 (3.9%)	0.4721 (25.6%)	0.3936 (4.7%)	0.3965 (5.5%)
1.10	0.3303	0.3523 (6.7%)	0.3523 (6.7%)	0.4556 (37.9%)	0.3547 (7.4%)	0.3580 (8.4%)
1.15	0.3304	0.3617 (9.5%)	0.3616 (9.4%)	0.4750 (43.8%)	0.3644 (10.3%)	0.3678 (11.3%)
1.20	0.3365	0.3779 (12.3%)	0.3779 (12.3%)	0.4995 (48.4%)	0.3811 (13.2%)	0.3846 (14.3%)
1.25	0.3436	0.3959 (15.2%)	0.3959 (15.2%)	0.5254 (52.9%)	0.3998 (16.4%)	0.4033 (17.4%)
1.30	0.3508	0.4145 (18.2%)	0.4144 (18.1%)	0.5517 (57.3%)	0.4191 (19.5%)	0.4227 (20.5%)

* See Appendix.

† Equations 24(a–d).

Isotropic, $E = 14$ GN/m^2, $\nu = 0.3$, $\ell/r_2 = 10$; Critical pressure, $\bar{p} = pr_2^3/(Eh^3)$; Fixed ends, $n = 2$, $m = 1$.

$$Q_{DS} = \frac{h^2}{12R^2(1-\nu^2)}(\tilde{m}^2+n^2)^2(\tilde{m}^2+n^2-1)^2+\tilde{m}^4. \tag{24d}$$

Again, a distinct eigenvalue corresponds to each pair of the positive integers m and n, the critical load being for the pair that renders the lowest eigenvalue.

Table 4 gives the predictions of the different isotropic shell theories for $\ell/r_2 = 10$, in comparison with the elasticity one. It is clearly seen that all shell theories predict higher critical values than the elasticity solution, the percentage increase being larger with thicker shells. However, both the direct Flügge and Danielson and Simmonds expressions predict critical loads much closer to the elasticity value than the direct Donnell expression. These were also very close to the ones predicted by the more involved, non-shallow Donnell and

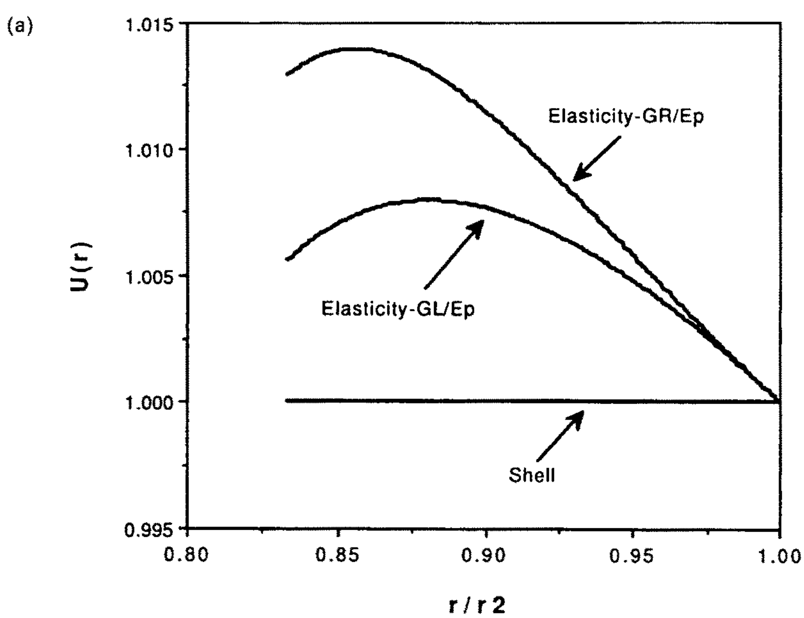

Fig. 2. (a) Eigenfunction $U(r)$ versus normalized radial distance r/r_2, for the two orthotropic cases [shell theory would have a constant value throughout, $U(r) = 1$ for all cases]. (b) Eigenfunction $V(r)$ versus normalized radial distance r/r_2 from the elasticity solution and the Donnell shell theory, which would show linear variation. The results are for the graphite/epoxy orthotropic case. (c) Eigenfunction $W(r)$ versus normalized radial distance r/r_2, from the elasticity solution and the Donnell shell theory (the latter has a linear variation). The results are for the graphite/epoxy orthotropic case.

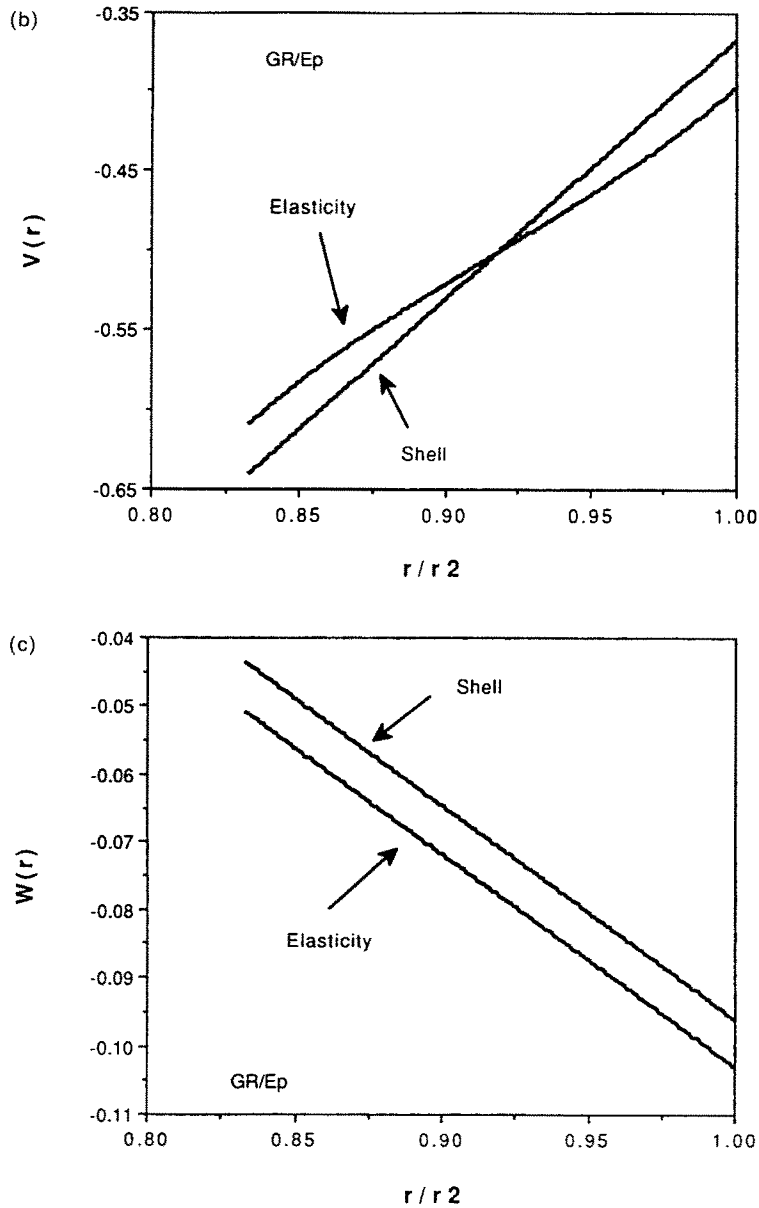

Fig. 2—(*continued*).

Timoshenko theories. A comparison of the data from all four Tables shows that for isotropic materials the degree of non-conservatism of the shell theories is much lower.

It should also be mentioned that the elasticity results of Tables 1 and 3 for the glass/epoxy material that were produced through the present formulation, which was based on assuming general, non-planar equilibrium modes, are very close to the results from the earlier simplified formulation of Kardomateas (1993a), which was based on plane equilibrium modes, i.e. a ring assumption.

Finally, to obtain more insight into the displacement field, Figs 2(a,b,c) show the variation of $U(r)$, $V(r)$, and $W(r)$, which define the eigenfunctions, for $r_2/r_1 = 1.20$, $\ell/r_2 = 10$, as derived from the present elasticity solution, and in comparison with the Donnell shell theory assumptions of constant $U(r)$, and linear $V(r)$ and $W(r)$. These values have been normalized by assigning a unit value for U at the outside boundary $r = r_2$.

These plots illustrate graphically the deviation of U from constant, and the deviation of V and W from linearity. Although the Donnell shell theory eigenfunction has been plotted for $V(r)$ and $W(r)$, the Timoshenko theory lines would nearly coincide with the latter. Notice that the distribution of $U(r)$ for the graphite/epoxy case shows the biggest deviation from the constant U value, shell theory assumption; the bifurcation load for this case shows also the biggest deviation from the shell theory predictions. In general, Figs 2(a) and (b) are similar to Figs 3 and 4 in Kardomateas (1993a) which were based on a ring approximation and glass/epoxy material. However, Fig. 2(a) of the present paper illustrates in addition the difference between the strongly orthotropic graphite/epoxy and the moderately orthotropic glass/epoxy. Furthermore Fig. 2(c) shows that both shell and elasticity give essentially a linear variation for $W(r)$.

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REFERENCES

Brush, D. O. and Almroth, B. O. (1975). *Buckling of Bars, Plates and Shells*. McGraw-Hill, New York.
Danielson, D. A. and Simmonds, J. G. (1969). Accurate buckling equations for arbitrary and cylindrical elastic shells. *Int. J. Engng Sci.* **7**, 459–468.
Donnell, L. H. (1933). Stability of thin-walled tubes under torsion. *NACA Rep.* 479.
Flügge, W. (1960). *Stresses in Shells*, pp. 426–432. Springer, Heidelberg.
Kardomateas, G. A. (1993a). Buckling of thick orthotropic cylindrical shells under external pressure. *J. Appl. Mech.* (ASME) **60**, 195–202.
Kardomateas, G. A. (1993b). Stability loss in thick transversely isotropic cylindrical shells under axial compression. *J. Appl. Mech.* (ASME) **60**, 506–513.
Kardomateas, G. A. (1993c). Bifurcation of equilibrium in thick orthotropic cylindrical shells under axial compression. *Proceedings Ninth International Conference on Composite Materials* (ICCM/9), Vol. IV, pp. 422–429, also in press, *J. Appl. Mech.* (ASME).
Lekhnitskii, S. G. (1963). *Theory of Elasticity of an Anisotropic Elastic Body*. Holden Day, San Francisco.
Librescu, L. (1975). *Elastostatics and Kinetics of Anisotropic and Heterogeneous Shell-Type Structures*. Nordhoff International, Leyden.
Press, W. H., Flannery, B. P., Teukolsky, S. A. and Vetterling, W. T. (1989). *Numerical Recipes*. Cambridge University Press, Cambridge.
Reddy, J. N. and Liu, C. F. (1985). A higher-order shear deformation theory of laminated elastic shells. *Int. J. Engng Sci.* **23**(3), 319–330.
Simitses, G. J., Shaw, D. and Sheinman, I. (1985). Stability of cylindrical shells by various nonlinear shell theories. *Z. Angew. Math. u. Mech.* (ZAMM) **65**(3), 159–166.
Timoshenko, S. P. and Gere, J. M. (1961). *Theory of Elastic Stability*. McGraw-Hill, New York.
Whitney, J. M. and Sun, C. T. (1974). A refined theory for laminated anisotropic cylindrical shells. *J. Appl. Mech.* **41**(2), 471–476.

APPENDIX: EIGENVALUES FROM NON-SHALLOW DONNELL AND TIMOSHENKO SHELL THEORIES

In the shell theory formulation, the mid-thickness ($r = R$) displacements are in the form:

$$u_1 = U_0 \cos n\theta \sin \lambda z, \quad v_1 = V_0 \sin n\theta \sin \lambda z, \quad w_1 = W_0 \cos n\theta \cos \lambda z,$$

where U_0, V_0, W_0 are constants.

The equations for the non-shallow (or non-simplified) Donnell shell theory are (Brush and Almroth, 1975):

$$RN_{z,z} + N_{z\theta,\theta} = 0,$$

$$RN_{z\theta,z} + N_{\theta,\theta} + \frac{M_{\theta,\theta}}{R} + M_{z\theta,z} = 0,$$

$$N_\theta - RN_z^0 u_{,zz} - RM_{z,zz} - \frac{M_{\theta,\theta\theta}}{R} - 2M_{z\theta,z\theta} + N_\theta^0 \beta_{\theta,\theta} + p(v_{,\theta} + u) = 0$$

where $R\beta_\theta = v - u_{,\theta}$. The Timoshenko shell theory (Timoshenko and Gere, 1961) has the additional term $-N_\theta^0(v_{,\theta z} + u_{,z})$ in the first equation, and the additional term $RN_z^0 v_{,zz}$ in the second equation. We have denoted by R the mean shell radius and by p the absolute value of the external pressure. Notice that for loading under external pressure p, $N_{z\theta}^0 = 0$ and $N_\theta^0 = -pR$ and if the pressure from the end caps is included, $N_z^0 = -pR/2$. For the case of a shell with fixed ends, $N_z^0 = 0$.

In terms of the "equivalent property" constants

$$C_{22} = E_2h/(1-\nu_{23}\nu_{32}), \quad C_{33} = E_3h/(1-\nu_{23}\nu_{32}),$$

$$C_{23} = \frac{E_3\nu_{23}h}{1-\nu_{23}\nu_{32}}, \quad C_{44} = G_{23}h, \quad D_{ij} = C_{ij}\frac{h^2}{12},$$

the coefficient terms in the homogeneous equations system that gives the eigenvalues are:

$$\alpha_{11} = C_{23}\lambda; \quad \alpha_{12} = (C_{23}+C_{44})n\lambda; \quad \alpha_{13} = -(C_{33}R\lambda^2+C_{44}n^2/R),$$

$$\alpha_{21} = -\left(\frac{C_{22}}{R}+\frac{D_{22}n^2}{R^3}+\frac{D_{23}\lambda^2}{R}+2\frac{D_{44}\lambda^2}{R}\right)n,$$

$$\alpha_{22} = -\left(\frac{C_{22}n^2}{R}+C_{44}R\lambda^2+\frac{D_{22}n^2}{R^3}+2\frac{D_{44}\lambda^2}{R}\right), \quad \alpha_{23} = (C_{23}+C_{44})n\lambda,$$

$$\alpha_{31} = \frac{C_{22}}{R}+\frac{D_{22}n^4}{R^3}+2\frac{D_{23}\lambda^2n^2}{R}+D_{33}\lambda^4R+4\frac{D_{44}\lambda^2n^2}{R},$$

$$\alpha_{32} = \left(\frac{C_{22}}{R}+\frac{D_{22}n^2}{R^3}+\frac{D_{23}\lambda^2}{R}+4\frac{D_{44}\lambda^2}{R}\right)n, \quad \alpha_{33} = -C_{23}\lambda.$$

Notice that in the above formulas we have used the curvature expression $\kappa_{z\theta} = 2(v_{,z}-u_{,z\theta})/R$ for both theories.

Then the linear homogeneous equations system that gives the eigenvalues for the Timoshenko shell formulation for the case of end caps is:

$$(\alpha_{11}+pR\lambda)U_0+(\alpha_{12}+pRn\lambda)V_0+\alpha_{13}W_0 = 0, \tag{A1}$$

$$\alpha_{21}U_0+\left(\alpha_{22}+p\frac{R^2\lambda^2}{2}\right)V_0+\alpha_{23}W_0 = 0, \tag{A2}$$

$$\left[\alpha_{31}-p\frac{R^2\lambda^2}{2}-p(n^2-1)\right]U_0+\alpha_{32}V_0+\alpha_{33}W_0 = 0. \tag{A3}$$

For the Donnell shell formulation, the additional term in the coefficient of V_0 in eqn (A2) is omitted, i.e. the coefficient of V_0 is only α_{22} and the additional terms in the coefficients of U_0 and V_0 in eqn (A1) are also omitted, i.e. the coefficient of U_0 is only α_{11} and the coefficient of V_0 is only α_{12}. For the simpler case of a cylindrical shell with fixed ends, the terms $pR^2\lambda^2/2$ are omitted in the second and third equations. The eigenvalues are naturally found by equating to zero the determinant of the coefficients of U_0, V_0 and W_0.